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# TWO TRANSITIVE MINKOWSKI PLANES

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Two transitive Minkowski planes[*)]

by

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ABSTRACT

In this paper we show that all Minkowski planes which have the property that for any two pairs of nonparallel points there is an automorphism mapping one pair to the other, are known.

KEY WORDS & PHRASES: *Minkowski plane, projective plane, affine plane, automorphism group*

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*) This report will be submitted for publication elsewhere.

## 1. INTRODUCTION

All known finite inversive planes have a two-transitive group of automorphisms. Conversely, every inversive plane admitting an automorphism group which is two-transitive on the points, is of known type (cf. [8]).

For Minkowski planes the situation is quite similar. All known finite Minkowski planes have an automorphism group acting two-transitively on nonparallel points. In this note we shall show that this property is characteristic for the known Minkowski planes. More precisely, we shall prove the following theorem.

THEOREM. *Let* $\mathcal{M}$ *be a finite Minkowski plane of odd order* n, *and suppose that* $\mathcal{M}$ *admits an automorphism group* $\Gamma$ *acting two-transitively on nonparallel points. Then* n *is a prime power,* $\mathcal{M} \simeq \mathcal{M}(n,\phi)$ *for some field automorphism* $\phi$ *of* GF(n), *and* $\Gamma$ *contains* $PSL(2,n) \times PSL(2,n)$.

For a definition of $\mathcal{M}(n,\phi)$ see Section 2. As Minkowski planes of even order n only exist for n a power of 2, and are unique for given order $n = 2^a$, this result completes the classification of the Minkowski planes with an automorphism group acting two-transitively on nonparallel points.

## 2. DEFINITIONS, NOTATION AND BASIC RESULTS

Let M be a set of *points* and $\mathcal{L}^+$, $\mathcal{L}$, $\mathcal{C}$ three collections of subsets of M. The elements of $\mathcal{L} := \mathcal{L}^+ \cup \mathcal{L}^-$ are called *lines* or *generators*, the elements of $\mathcal{C}$ are called *circles*. We say that $\mathcal{M} = (M,\mathcal{L}^+,\mathcal{L}^-,\mathcal{C})$ is a *Minkowski plane* if the following axioms are satisfied (cf. [7]).

(M1): $\mathcal{L}^+$ and $\mathcal{L}^-$ are partitions of M.

(M2): $|\ell^+ \cap \ell^-| = 1$ for all $\ell^+ \in \mathcal{L}^+$, $\ell \in \mathcal{L}^-$.

(M3): Given any three points no two on a line, there is a unique circle passing through these three points.

(M4): $|\ell \cap c| = 1$ for all $\ell \in \mathcal{L}$, $c \in \mathcal{C}$.

(M5): Given a circle c, a point $P \in c$ and a point $Q \notin c$, P and Q not on one line, there is a unique circle d such that $P,Q \in d$ and $c \cap d = \{P\}$.

Two points P and Q are called *plus-parallel* (notation $P \parallel_+ Q$) if P and Q are on a line of $\mathcal{L}^+$, *minus-parallel* (notation $P \parallel_- Q$) if P and Q are on a line of $\mathcal{L}^-$. *Parallel* (notation $P \parallel Q$) means either $P \parallel_+ Q$ or $P \parallel_- Q$. For $P \in M$ we denote by $[P]_+$ (resp. $[P]_-$) the unique line in $\mathcal{L}^+$ (resp. $\mathcal{L}^-$) incident with P. If P, Q and R are (distinct) nonparallel points, then we denote by (P,Q,R) the unique circle containing P, Q and R. Two circles c and d *touch* in a point P if $c \cap d = \{P\}$.

We shall only consider finite Minkowski planes, i.e. Minkowski planes with a finite number of points. For finite Minkowski planes (M6) is a consequence of the other axiom (see [7]). It is easily seen that $|\mathcal{L}^+| = |\mathcal{L}^-| = |\ell| = |c| =: n+1$ for all $\ell \in \mathcal{L}$, $c \in \mathcal{C}$. The integer n is called the order of the Minkowski plane. Fix a point P and put

$$M_P := M \setminus ([P]_+ \cup [P]_-),$$

$$L_P := \{c^* \mid c \in \mathcal{C},\ P \in c\} \cup \{\ell^* \mid \ell \in \mathcal{L} \setminus \{[P]_+,[P]_-\}\},$$

where the * indicates that we have removed the point that the circle or line has in common with $[P]_+ \cup [P]_-$. Then $\mathcal{M}_P := (M_P, L_P)$ is an affine plane with point set $M_P$ and line set $L_P$ (see e.g. [7]). The projective plane associated with $\mathcal{M}_P$ will be denoted by $\tilde{\mathcal{M}}_P$. We call $\mathcal{M}_P$ the *derived plane* with respect to the point P.

Following BENZ [2] we sketch the close relationship between finite Minkowski planes and sharply triply transitive sets of permutations. Let $\Omega$ be a finite set, $|\Omega| = n+1$, and let G be a subset of $\mathrm{Sym}(\Omega)$, the symmetric group on $\Omega$, acting sharply triply transitively on $\Omega$. Define

$$M := \Omega \times \Omega,$$

$$\mathcal{L}^+ := \{\{(\alpha,\beta) \mid \alpha \in \Omega\} \mid \beta \in \Omega\},$$

$$\mathcal{L}^- := \{\{(\alpha,\beta) \mid \beta \in \Omega\} \mid \alpha \in \Omega\},$$

$$\mathcal{C} := \{\{(\alpha,\alpha^g) \mid \alpha \in \Omega\} \mid g \in G\}.$$

Then $\mathcal{M} := (\Omega,G) := (M,\mathcal{L}^+,\mathcal{L}^-,\mathcal{C})$ is a Minkowski plane of order n. Conversely, every Minkowski plane can be obtained in this way.

Two Minkowski planes $\mathcal{M} = (\Omega,G) = (M,\mathcal{L}^+,\mathcal{L}^-,\mathcal{C})$ and $\mathcal{M}' = (\Omega',G') = (M',\mathcal{L}^{+'},\mathcal{L}^{-'},\mathcal{C}')$ are said to be *isomorphic* if there is a bijection $s: M \to M'$ such that

$$\mathcal{L}^s = \mathcal{L}' \quad \text{and} \quad \mathcal{C}^s = \mathcal{C}'.$$

Either $(\mathcal{L}^+)^s = \mathcal{L}^{+'}$ and $(\mathcal{L}^-)^s = \mathcal{L}^{-'}$ or $(\mathcal{L}^+)^s = \mathcal{L}^{-'}$ and $(\mathcal{L}^-)^s = \mathcal{L}^{+'}$. In the first case s is called a *positive isomorphism*, in the second case a *negative isomorphism*. If s is a positive isomorphism then there exist bijections a,b: $\Omega \to \Omega'$ such that $(\alpha,\beta)^s = (\alpha^a,\beta^b)$ for all $\alpha,\beta \in \Omega$, and $a^{-1}Gb = G'$. If s is a negative isomorphism then there exist bijections a,b: $\Omega \to \Omega'$ such that $(\alpha,\beta)^s = (\beta^b,\alpha^a)$ and $b^{-1}G^{-1}a = G'$. It follows that we may assume w.l.o.g. that G contains the indentity permutation on $\Omega$.

A (*positive, negative*) *automorphism* of a Minkowski plane $\mathcal{M}$ is a (positive, negative) isomorphism of $\mathcal{M}$ onto itself. The automorphism group $\mathrm{Aut}(\Omega,G) \leq \mathrm{Sym}(\Omega\times\Omega)$ of the Minkowski plane $(\Omega,G)$ is given by

$$\mathrm{Aut}(\Omega,G) = \{(a,b) \in \mathrm{Sym}(\Omega) \times \mathrm{Sym}(\Omega) \mid a^{-1}Gb = G\} \cup$$

$$\cup \{(a,b) \in \mathrm{Sym}(\Omega) \times \mathrm{Sym}(\Omega) \mid a^{-1}Gb = G^{-1}\}\tau$$

where $\tau \in \mathrm{Sym}(\Omega\times\Omega)$ is defined by $(\alpha,\beta)^\tau = (\beta,\alpha)$ for all $(\alpha,\beta) \in \Omega\times\Omega$.

We conclude this section by describing all known finite Minkowski planes (cf. [11]). Let q be a prime power and let $\phi$ be a field automorphism of GF(q). We shall denote by $\mathcal{M}(q,\phi)$ the Minkowski plane $(\Omega,G)$ with $\Omega = PG(1,q)$, the projection line of order q, and with G the subset of $\mathrm{Sym}(\Omega)$ consisting of the permutations

$$x \mapsto \frac{ax+b}{cx+d}, \qquad ad-bc = \text{a non-zero square of } GF(q),$$

and

$$x \mapsto \frac{ax^\phi+b}{cx^\phi+d}, \qquad ad-bc = \text{a nonsquare of } GF(q).$$

Of course, if q is even we always get $G = PSL(2,q)$, and it can be shown that these are the only Minkowski planes of even order (see [6]). For q odd, G is a group if and only if $\phi^2 = 1$ (see [9]), and nonisomorphic Minkowski planes of the same order q can exist. Notice that $M(q,\phi)$ has an automorphism group containing $PSL(2,q) \times PSL(2,q)$ which is two-transitive on nonparallel points, i.e. if P, Q, P', Q' are points such that $P \nparallel Q$ and $P' \nparallel Q'$, then there is an automorphism g satisfying $P^g = P'$ and $Q^g = Q'$.

## 3. PROOF OF THEOREM

For the proof of our theorem we require a number of lemmas. The first lemma shows that we can assume without loss of generality that an automorphism group which is two-transitive on nonparallel points contains positive automorphisms only.

<u>LEMMA 1</u>. *Let* $M = (M, L^+, L^-, C)$ *be a Minkowski plane of odd order* n *and let* $\Gamma^*$ *be a group of automorphisms of* $M$ *two-transitive on nonparallel points. Then* $\Gamma := \Gamma_{L^+} = \Gamma_{L^-}$ *is also two-transitive on nonparallel points* ($\Gamma_{L^+}$ *is the setwise stabilizer of* $L^+$ *in* $\Gamma^*$).

<u>PROOF</u>. Let P and Q be two points, $P \nparallel Q$. Then

$$[\Gamma_P : \Gamma_{PQ}] = [\Gamma_P^* : \Gamma_{PQ}^*][\Gamma_{PQ}^* : \Gamma_{PQ}][\Gamma_P^* : \Gamma_P]^{-1},$$

Now $[\Gamma_P^* : \Gamma_{PQ}^*] = |M_P| = n^2$ (as before $M_P = M \setminus ([P]_+ \cup [P]_-) = \{R \mid R \nparallel P\}$), and $[\Gamma_{PQ}^* : \Gamma_{PQ}], [\Gamma_P^* : \Gamma_P] \in \{1,2\}$ since $[\Gamma^* : \Gamma] \in \{1,2\}$. Since n is odd it follows that $[\Gamma_P : \Gamma_{PQ}] = n^2$, i.e. $\Gamma_P$ is transitive on $M_P$. Hence $\Gamma$ is two-transitive on nonparallel points. □

From now on $M = (M, L^+, L^-, C) = (\Omega, G)$ is a Minkowski plane of odd order $n \geq 5$ with a group $\Gamma$ of positive automorphisms acting two-transitively on nonparallel points. (For n = 3 the theorem follows readily from [4].) We denote by $\Gamma(L^\varepsilon)$ the subgroup of $\Gamma$ fixing all lines of $L^\varepsilon$, $\varepsilon = +,-$. Notice that $\Gamma(L^{-\varepsilon})$ has a faithful representation on the (n+1) lines of $L^\varepsilon$, $\varepsilon = +,-$.

LEMMA 2. *If* $\Gamma(L^{\varepsilon})$ *contains* PSL(2,n) *for* $\varepsilon = +$ *or* $-$, *then* $M \simeq M(n,\phi)$ *for some* $\phi \in \mathrm{Aut}(GF(n))$ *and* $\Gamma$ *contains* $PSL(2,n) \times PSL(2,n)$.

PROOF. For convenience we take $\varepsilon = -1$. As a permutation group on $M = \Omega \times \Omega$, $\Gamma$ consists of permutations $(a_\gamma, b_\gamma) \in \mathrm{Sym}(\Omega) \times \mathrm{Sym}(\Omega)$ satisfying $a_\gamma^{-1} G b_\gamma = G$, $\gamma \in \Gamma$. Clearly, $\Sigma \leq \Gamma(L^-)$ is equivalent to $a_\sigma = 1$ for all $\sigma \in \Sigma$. Hence $B := \{b_\sigma \mid \sigma \in \Sigma\}$ is a subgroup of $\mathrm{Sym}(\Omega)$ satisfying $GB = G$. Therefore G consists of a number of cosets of B, in particular $B \subseteq G$ since we are assuming that $1 \in G$. If $\Sigma \simeq B = G_1 := PSL(2,n)$ then $G = G_1 \cup \phi G_2$ for some $\phi \in \mathrm{Sym}(\Omega)$ where $G_2 := PGL(2,n)\backslash G_1$ ($|G_1| = \frac{1}{2}(n+1)n(n-1)$ and $|G| = |C| = (n+1)n(n-1)$). Viewing $\Omega$ as the projective line $GF(n) \cup \{\infty\}$ in the appropriate way, we claim that we may take $\phi \in \mathrm{Aut}(GF(n))$. Let x, y and z be three distinct points of $\Omega$. Since G is sharply triply transitive on $\Omega$, there exists a $g \in G$ such that $x^\phi = x^g$, $y^\phi = y^g$ and $z^\phi = z^g$. Suppose $g \in \phi G_2$, i.e. $g = \phi g_2$ for some $g_2 \in G_2$. Then $x^\phi = (x^\phi)^{g_2}$, $y^\phi = (y^\phi)^{g_2}$, $z^\phi = (z^\phi)^{g_2}$, and we get the contradiction $1 = g_2 \in G_2$.

We have shown: for any three distinct $x,y,z \in \Omega$ there is a $g_1 \in G_1$ such that $x^\phi = x^{g_1}$, $y^\phi = y^{g_1}$ and $z^\phi = z^{g_1}$. It follows that we may assume without loss of generality that $\phi$ fixes 0, 1 and $\infty$. If we do so it also follows that

$$\frac{x^\phi - y^\phi}{x-y} \text{ is a square in } GF(n) \text{ for all } x,y \in GF(n),\ x \neq y,$$

for $g_1 \in G_1$ determined by $x^\phi = x^g$, $y^\phi = y^g$, $\infty^\phi = \infty = \infty^g$ is the permutation $(\xi \mapsto ((x^\phi - y^\phi)/(x-y))(\xi - y) + y^\phi) \in G_1$. By a theorem of BRUEN and LEVINGER (see [3]) it follows that $\phi \in \mathrm{Aut}(GF(n))$. It remains to show that $\Gamma(L^+)$ also contains PSL(2,n). Let $\gamma = (a_\gamma, b_\gamma) \in \Gamma$, then $a_\gamma^{-1} b_\gamma \in a_\gamma^{-1} G b_\gamma = G \subseteq P\Gamma L(2,n)$. Hence,

$$G_1^{a_\gamma} \subseteq G^{a_\gamma} = a_\gamma^{-1} G a_\gamma = a_\gamma^{-1}(a_\gamma G b_\gamma^{-1}) a_\gamma = G(a_\gamma^{-1} b_\gamma)^{-1} \subseteq P\Gamma L(2,n).$$

Since $G_1^{a_\gamma}$ is a two-transitive subgroup of $P\Gamma L(2,n)$, $G_1^{a_\gamma}$ contains $G_1$ so $G_1^{a_\gamma} = G_1$. Therefore $a_\gamma \in P\Gamma L(2,n)$. Now $\{a_\gamma \mid \gamma \in \Gamma\}$ is a two-transitive subgroup of $P\Gamma L(2,n)$, hence contains $G_1$. Since $a_\gamma^{-1} b_\gamma \in G = G_1 \cup \phi G_2$ and $a_\gamma^{-1} G_1 b_\gamma = G_1^{a_\gamma}(a_\gamma^{-1} b_\gamma) = G_1(a_\gamma^{-1} b_\gamma)$ either $a_\gamma^{-1} G_1 b_\gamma = G_1$ or $a_\gamma^{-1} G_1 b_\gamma = \phi G_2$. Since $G_1$ does not contain a subgroup of index 2, $\{a_\gamma \mid \gamma \in \Gamma,\ a_\gamma^{-1} G_1 b_\gamma = G_1\}$ contains $G_1$. Let $a \in G_1$, then there is a $\gamma \in \Gamma$ such that $\gamma = (a,b)$, $a^{-1} G_1 b = G_1$. Since $a \in G_1$

also $b \in G_1$. Hence $(1,b^{-1}) \in \Gamma$ and so $(a,1) = (a,b)(1,b^{-1}) \in \Gamma(L^+)$. □

LEMMA 3. *Let* $\varepsilon$ *be* + *or* −. *If* $\Sigma \le \Gamma(L^{\varepsilon})$ *is transitive on* $L^{-\varepsilon}$ *and* $\Sigma_{\ell,m} = 1$ *for distinct* $\ell,m \in L^{-\varepsilon}$, *then* $|\Sigma_{\ell}| \le 3$ *for all* $\ell \in L^{-\varepsilon}$.

PROOF. Since $\Sigma \le \Gamma(L^{\varepsilon})$, G contains a group $H \simeq \Sigma$ (as permutation groups). Since $\Sigma_{\ell,m} = 1$ for distinct $\ell,m \in L^{-\varepsilon}$, $H_{\alpha,\beta} = 1$ for distinct $\alpha,\beta \in \Omega$. It follows that the circles corresponding to the elements of H cannot intersect each other in two points. It is not hard to see that $|\Sigma_{\ell}| = |H_{\alpha}| > 3$ implies that we can find four circles $c_1$, $c_2$, $c_3$ and d such that the $c_i$ touch each other in a point P not on d and such that the $c_i$ touch d in three distinct points of d. This, however, means that in the projective plane $\widetilde{M}_P$ the oval corresponding to d has three tangents passing through a common point. As n, the order of $\widetilde{M}_P$, is odd, this is a contradiction. □

LEMMA 4. *Let* $\varepsilon$ *be* + *or* −. *If* $\Gamma(L^{\varepsilon})$ *is two-transitive on* $L^{-\varepsilon}$, *then* n *is a prime power,* $M \simeq M(n,\phi)$ *for some* $\phi \in \mathrm{Aut}(GF(n))$ *and* $\Gamma$ *contains* $PSL(2,n) \times PSL(2,n)$.

PROOF. As G is sharply triply transitive on $\Omega$, $\Gamma(L^{\varepsilon})_{\ell,m,n} = 1$ for distinct lines $\ell,m,n \in L^{-\varepsilon}$. By results 4.3.27 (p. 197) of [5], either $\Gamma(L^{\varepsilon})$ contains a sharply two-transitive subgroup, or $\Gamma(L^{\varepsilon})$ contains $PSL(2,n)$ as a normal subgroup of index $\le 2$, or $\Gamma(L^{\varepsilon}) \simeq Sz(\sqrt{n})$ with n power of 2. Since n is odd, the last alternative cannot occur. The first alternative is impossible by Lemma 3. Lemma 2 now completes the proof. □

LEMMA 5. *If* $\Gamma(L^{\varepsilon})$ *contains a nontrivial element fixing two lines of* $L^{-\varepsilon}$ ($\varepsilon$ = + *or* −), *then* n *is a prime power,* $M \simeq M(n,\phi)$ *for some* $\phi \in \mathrm{Aut}(GF(n))$ *and* $\Gamma$ *contains* $PSL(2,n) \times PSL(2,n)$.

PROOF. Suppose $1 \ne \gamma \in \Gamma(L^{\varepsilon})$ fixes $\ell,m \in L^{-\varepsilon}$, $\ell \ne m$. We may assume that $\gamma$ has prime order. As remarked in the proof of Lemma 4, $\gamma$ fixes no other lines of $L^{-\varepsilon}$ besides $\ell$ and m. Since $\Gamma(L^{\varepsilon})$ is a normal subgroup of $\Gamma$, $\langle \gamma^{\alpha} \mid \alpha \in \Gamma_{\ell} \rangle \le \Gamma(L^{\varepsilon})$. By a result of GLEASON (see [5], 4.3.15, p. 191), it follows that $\langle \gamma^{\alpha} \mid \alpha \in \Gamma_{\ell} \rangle$ is transitive on $L^{-\varepsilon} \setminus \{\ell\}$. Hence $\langle \gamma^{\alpha} \mid \alpha \in \Gamma \rangle$ is two-transitive on $L^{-\varepsilon}$. Now apply Lemma 4. □

From the foregoing lemmas it is clear that our main objective will be to show that $\Gamma(L^{\varepsilon})$ is nontrivial. For this it is necessary first to investigate how $\Gamma$ acts on $C$ and how $\Gamma_P$ acts on $M_P$, $P \in M$. Define a *pencil* to be any maximal set of mutually tangent circles through a common point P, called the *carrier* of the pencil. Thus the pencils with given carrier P are essentially identical with parallel classes of lines in the affine plane $M_P$. Every pencil contains n circles. Every point is carrier of n-1 pencils.

LEMMA 6. *For every point* P *and pencil* $P$ *with carrier* P, $\Gamma_{P,P}$ *is transitive on the* n *circles of* $P$.

PROOF. Since $\Gamma$ is two-transitive on nonparallel points, $\Gamma_P$ is transitive on the points of $M_P$. By Theorem 3 of [13] we are done. □

Thus, if circles c and d touch, then there exists $\gamma \in \Gamma$ such that $c^{\gamma} = d$. This shows that every $\Gamma$-orbit on $C$ consists of a number of components of the *touch-graph* defined on $C$ by: $c,d \in C$ are *adjacent* iff c and d touch.

LEMMA 7. *The touch-graph has* 1 *or* 2 *components. If it has* 2 *components, then each conponent contains* $\frac{1}{2}(n+1)n(n-1)$ *circles and every point is incident with* $\frac{1}{2}n(n-1)$ *circles of each component.*

PROOF. Let $c_1$, $c_2$ and $c_3$ be three distinct circles and P a point, $P \notin c_1,c_2,c_3$. The ideal line of the affine plane $M_P$ consists of the ideal points (i.e. parallel classes of $M_P$) $L^+\backslash\{[P]_+\}$, $L^-\backslash\{[P]_-\}$ and the (n-1) pencils with carrier P. The circles $c_1$, $c_2$ and $c_3$ correspond to ovals intersecting the ideal line in $L^+\backslash\{[P]_+\}$ and $L^-\backslash\{[P]_-\}$. Thus, since n is odd, for each $c_i$ there are $\frac{1}{2}(n-1)$ ideal points which are exterior with respect to $c_i$ (i.e. are the point of intersection of two tangents of $c_i$) and $\frac{1}{2}(n-1)$ ideal points which are interior with respect to $c_i$. This shows that at least two of $c_1$, $c_2$ and $c_3$ have an exterior point on the ideal line in common, hence are in the same component of the tough-graph. Therefore, the number of components is at most 2. If there are 2 components and $c_1$ and $c_2$, say, are in distinct components, then the ideal points corresponding to the pencils fall into two classes: $\frac{1}{2}(n-1)$ are exterior with respect to $c_1$ and the other $\frac{1}{2}(n-1)$ are exterior with respect to $c_2$. Hence P is incident with $\frac{1}{2}n(n-1)$ circles of each component, and an easy counting argument shows that each component contains $\frac{1}{2}(n+1)n(n-1)$ circles. □

REMARK. The touch-graph of $\mathcal{M}(q,\phi)$, q odd, actually has two components.

By Lemmas 6 and 7, if t is the number of $\Gamma$-orbits on $\mathcal{C}$, $t \in \{1,2\}$ and $[\Gamma:\Gamma_c] = t^{-1}(n+1)n(n-1)$ for all $c \in \mathcal{C}$. Using this result we can show the transitivity properties stated in the next lemma.

LEMMA 8.

(i) *If* c *is a circle, then* $\Gamma_c$ *is two-transitive on* c.

(ii) *If* P *is a point, then* $\Gamma_P$ *has* t *orbits of length* $t^{-1}(n-1)$ *on the pencils with carrier* P.

(iii) *If* P *and* Q *are distinct point,* $P \not\parallel Q$*, then* $\Gamma_{P,Q}$ *has* t *orbits of length* $t^{-1}(n-1)$ *on the circles containing* P *and* Q.

(iv) *If* P *and* Q *are distinct points of the circle* c*, then* $|\Gamma| = (n+1)^2 n^2 (n-1) t^{-1} |\Gamma_{P,Q,c}|$.

PROOF. Let P and Q be distinct points of the circle c, and let $\mathcal{P}$ be the pencil with carrier P containing c. Denote by s the number of pencils in the $\Gamma_P$-orbit containing $\mathcal{P}$. Then $[\Gamma_P:\Gamma_{P,\mathcal{P}}] = s$ and $[\Gamma_P:\Gamma_{P,c}] = ns$ by Lemma 6. Hence,

$$(n+1) \geq [\Gamma_c:\Gamma_{c,P}] = \frac{|\Gamma_c|}{|\Gamma|}\cdot\frac{|\Gamma|}{|\Gamma_P|}\cdot\frac{|\Gamma_P|}{|\Gamma_{P,c}|} = \frac{1}{t^{-1}(n+1)n(n-1)}\cdot(n+1)^2\cdot ns$$

$$= \frac{st(n+1)}{n-1} = st + \frac{2st}{n-1} .$$

Thus, $st = \frac{1}{2}(n-1)u$ with $u \in \mathbb{N}$, and so $(n+1) \geq [\Gamma_c:\Gamma_{c,P}] = \frac{1}{2}(n+1)u$, i.e. $u \in \{1,2\}$. As $s = \frac{1}{2}t^{-1}(n-1)u$ with $u,t \in \{1,2\}$ and n is odd, $(n,s) = 1$. Therefore it follows from

$$n \geq \frac{|\Gamma_{c,P}|}{|\Gamma_{c,P,Q}|} = \frac{|\Gamma_{P,c}|}{|\Gamma_P|}\cdot\frac{|\Gamma_P|}{|\Gamma_{P,Q}|}\cdot\frac{|\Gamma_{P,Q}|}{|\Gamma_{P,Q,c}|}$$

$$= \frac{1}{ns}\cdot n^2\cdot[\Gamma_{P,Q}:\Gamma_{P,Q,c}] = \frac{n}{s}[\Gamma_{P,Q}:\Gamma_{P,Q,c}]$$

that $[\Gamma_{c,P}:\Gamma_{c,P,Q}] = n$ and $[\Gamma_{P,Q}:\Gamma_{P,Q,c}] = s$. Now from $[\Gamma_{c,P}:\Gamma_{c,P,Q}] = n$ it follows that $\Gamma_{c,P}$ is transitive on $c\setminus\{P\}$, hence, since P was an arbitrary point of c, $\Gamma_c$ is two-transitive on c. Therefore $(n+1) = [\Gamma_c:\Gamma_{c,P}] = \frac{1}{2}(n+1)u$, so $u = 2$ and $s = t^{-1}(n-1)$. Finally,

$$|\Gamma| = \frac{|\Gamma|}{|\Gamma_P|}\cdot\frac{|\Gamma_P|}{|\Gamma_{P,Q}|}\cdot\frac{|\Gamma_{P,Q}|}{|\Gamma_{P,Q,c}|}\cdot|\Gamma_{P,Q,c}| = (n+1)^2 n^2 (n-1)t^{-1}\cdot|\Gamma_{P,Q,c}|$$

which proves (iv). □

LEMMA 9. *Let* P *be a point. If* $\Gamma_P$ *has odd order, then* n *is a power of a prime,* $M \simeq M(n,\phi)$ *for some* $\phi \in \mathrm{Aut}(GF(n))$ *and* $\Gamma$ *contains* $PSL(2,n) \times PSL(2,n)$.

PROOF. Fix a line $\ell \in L^+$ and let $\Delta \simeq \Gamma_\ell/(\Gamma(L^-) \cap \Gamma_\ell)$ be the permutation group on $\ell$ induced by $\Gamma_\ell$. As $\Gamma$ is two-transitive on the nonparallel points of $M$, $\Delta$ is two-transitive on $\ell$. As $\Gamma_P$ has odd order, $\Delta_P$ has odd order for all $P \in \ell$. By SATZ 1 of [1], either $\Delta$ is solvable or $\Delta$ contains PSL(2,n) as a normal subgroup. If $\Delta$ is solvable, then $\Delta$ is isomorphic to a subgroup of the group of semilinear transformations of a Galois field of characteristic 2, i.e. $n+1 = 2^a$ for some $a \in \mathbb{N}$ and $|\Delta| \mid (n+1)na$. If $\Delta$ contains PSL(2,n) as a normal subgroup, then $n = p^b$ for some prime p and $b \in \mathbb{N}$ and $\Delta$ is a subgroup of $P\Gamma L(2,n)$, i.e. $|\Delta| \mid (n+1)n(n-1)b$. By Lemma 8(iv), the order of $\Gamma_\ell$ is $(n+1)n^2\ (n-1)t^{-1}\cdot|\Gamma_{P,Q,c}|$. In both cases it follows from $n \geq 5$ that $|\Gamma(L^-) \cap \Gamma_\ell| = |\Gamma(L^-)_\ell| > 3$. Since $\Gamma(L^-) \trianglelefteq \Gamma$ and $\Gamma$ acts doubly transitively on $L^+$, $\Gamma(L^-)$ acts transitively on $L^+$. By Lemma 3 there exists a nontrivial element of $\Gamma(L^-)$ fixing two distinct lines of $L^+$. Lemma 5 now completes the proof. □

By the previous lemma we may assume from now on that $\Gamma_P$ has even order. More in particular, $\Gamma_P$ contains involutions. Since n is odd, every involution $\tau \in \Gamma_P$ either induces a homology of the projective plane $\widetilde{M}_P$ associated with the affine plane $M_P$, or the $\tau$-fixed points and lines of $\widetilde{M}_P$ constitute a Baer subplane of $\widetilde{M}_P$ (cf. [5], p. 172). Our next lemma deals with the case where $\Gamma_P$ contains a homology.

LEMMA 10. *Let* $P \in M$ *and suppose that* $\tau \in \Gamma_P$ *is an involution which, considered as a collineation of* $\widetilde{M}_P$*, is a homology. Then* n *is a prime power,* $M \simeq M(n,\phi)$ *for some* $\phi \in \mathrm{Aut}(GF(n))$ *and* $\Gamma$ *contains* $PSL(2,n) \times PSL(2,n)$. *If* $\Gamma_P$ *has even order and*

(i) n *is not a square, or*

(ii) $t = 1$ *(i.e.* $\Gamma$ *is transitive on* $C$*).*

*then* $\Gamma_P$ *contains homologies.*

PROOF. We distinguish two cases:

Case (a). The axis of $\tau$ is the ideal line of $M_P$. Now, since $\Gamma_P$ is transitive on $M_P$, $M_P$ is a translation plane and $\Gamma_P$ contains the full translation group of $M_P$ (see [5], p. 187, result 4.3.2). Let $\Sigma^{(P)}$ be the subgroup of $\Gamma_P$ consisting of those translations of $M_P$ which fix all lines of $L^-$. Then $\Sigma^{(P)}$ is transitive on $L^+\backslash\{[P]_+\}$, hence $\Sigma := \langle\Sigma^{(P)} \mid P \in M\rangle$ is two-transitive on $L^+$. Since $\Sigma \leq \Gamma(L^-)$ we are done by Lemma 4.

Case (b). The axis of $\tau$ is an affine line of $M_P$. Clearly, the axis of $\tau$ corresponds to a line $\ell \neq [P]_+$, $[P]_-$ of $M$, say $\ell \in L^+\backslash\{[P]_+\}$. Now $1 \neq \tau \in \Gamma(L^-)$ and $\tau$ fixes the two distinct lines $[P]_+$ and $\ell$ of $L^+$. By Lemma 5 we have completed the proof of our first claim.

The order of a Baer subplane of $M_P$ is $\sqrt{n}$. Hence, if n is not a square, every involution in $\Gamma_P$ acts as a homology of $\tilde{M}_P$. Suppose $t = 1$. Let $\Lambda$ be a Sylow 2-subgroup of $\Gamma_P$ and let $\tau$ be an involution in the center of $\Lambda$. Suppose the $\tau$-fixed points and lines of $\tilde{M}_P$ constitute a Baer subplane. The two ideal points of $M_P$ corresponding to $L^+$ and $L^-$ are fixed by $\Gamma_P$, and by Lemma 8(ii) $\Gamma_P$ is transitive on the remaining n-1 ideal points. Let $2^a \| (n-1)$. By [14], Theorem 3.4', every shortest $\Lambda$-orbit on these n-1 ideal points has length $2^a$. The ideal line of $M_P$ is fixed by $\tau$ and contains therefore, apart from the ideal points corresponding to $L^+$ and $L^-$, $\sqrt{n}-1$ fixed points. Since $\tau \in Z(\Lambda)$, $\Lambda$ permutes these $\sqrt{n}-1$ points. However, $2^b \| (\sqrt{n}-1)$ with $b < a$, contradicting the fact that each of these $\sqrt{n}-1$ points is in a $\Lambda$-orbit of shortest length $2^a$. □

For the proof of our main result we need one more definition and lemma.

DEFINITION. Suppose $M_1 \subseteq M$; $L_1^\varepsilon \subseteq L^\varepsilon$, $\varepsilon = +,-$; $C_1 \subseteq C$. Put $L_1^{\varepsilon *} := \{\ell \cap M_1 \mid \ell \in L_1^\varepsilon\}$, $\varepsilon = +,-$; $C_1^* := \{c \cap M_1 \mid c \in C_1\}$. If $M_1 := (M_1, L_1^{+*}, L_1^{-*}, C_1^*)$ is a Minkowski plane with the property that any two circles which touch in $M_1$, touch in $M$, then $M_1$ is called a *subplane* of $M$ (compare [5], p. 258).

LEMMA 11. *Let $\Delta$ be a group of positive automorphisms of $M$. Let $M_1$ be the set of points left fixed by $\Delta$; $L_1^+$ (resp. $L^-$) the set of lines of $L^+$ (resp. $L^-$) left fixed by $\Delta$; and $C_1$ the set of circles left fixed by $\Delta$. Then $M_1 := (M_1, L_1^{+*}, L_1^{-*}, C_1^*)$ is a subplane of $M$ if and only if $M_1$ contains, at least three* *mutually nonparallel points.*

PROOF. Straightforward verification. $\square$

We are now ready to prove our main result.

THEOREM. *Let* $\mathcal{M} = (M, \mathcal{L}^+, \mathcal{L}^-, \mathcal{C})$ *be a finite Minkowski plane of odd order* n, *and suppose that* $\mathcal{M}$ *admits an automorphism group* $\Gamma$ *two-transitive on nonparallel points. Then* n *is a prime power,* $\mathcal{M} \simeq \mathcal{M}(n,\phi)$ *for some* $\phi \in \mathrm{Aut}(GF(n))$ *and* $\Gamma$ *contains* $PSL(2,n) \times PSL(2,n)$.

PROOF. Suppose $\mathcal{M}$ is a counter example to the theorem of minimal order. By Lemma 1 we may assume that $\Gamma$ contains positive automorphisms only. By Lemma 9, $\Gamma_P$ has even order for all $P \in M$. By Lemma 10 every involution in $\Gamma_P$ has has $(\sqrt{n}+1)^2$ fixed points. Hence, if $\Lambda$ is a 2-subgroup of $\Gamma$ maximal with respect to fixing at least three mutually nonparallel points, $\Lambda \neq 1$. Let $\mathcal{M}_1 = (M_1, \mathcal{L}_1^{+*}, \mathcal{L}_1^{-*}, \mathcal{C}_1^*)$ be the subplane of $\mathcal{M}$ consisting of the $\Lambda$-fixed points, lines and circles of $\mathcal{M}$ of order $n_1$, say. Clearly $n_1$ is odd, and since $\Lambda \neq 1$ we have $n_1 < n$. We claim that $N_\Gamma(\Lambda)$, considered as an automorphism group of $\mathcal{M}_1$, acts two-transitively on the nonparallel points of $\mathcal{M}_1$. To see this, let $c \in \mathcal{C}_1$. Then $\Lambda \leq \Gamma_c$ and $\Lambda$, considered as a permutation group on c, is a 2-subgroup of $\Gamma_c$ maximal with respect to fixing at least three points of c. By Lemma 8(i), $\Gamma_c$ is two-transitive on c, hence $N_{\Gamma_c}(\Lambda)$ is two-transitive on $c^* := c \cap M_1$ (see [1], Lemma 3.3). Now let $A_1$, $A_2$ and $B_1$, $B_2$ be two pairs of nonparallel points of $\mathcal{M}_1$. If $A_i \nparallel B_j$, $i,j = 1,2$, and $c_1$ is the unique circle containing $A_2, B_1, B_2$, and $c_2$ is the unique circle containing $A_2, B_1, B_2$, then there is a $\gamma_1 \in N_{\Gamma_{c_1}}(\Lambda)$ and a $\gamma_2 \in N_{\Gamma_{c_2}}(\Lambda)$ such that $A_1^{\gamma_1} = A_2$, $A_2^{\gamma_1} = B_1$, $A_1^{\gamma_2} = B_1$, $B_1^{\gamma_2} = B_2$. Hence $\gamma = \gamma_1\gamma_2 \in N_\Gamma(\Lambda)$ satisfies $A_1^\gamma = B_1$ and $A_2^\gamma = B_2$. Repeated application of this result in case $A_i \parallel B_j$ for some i and j, proves our claim. Since $\mathcal{M}$ was supposed to be a minimal counter example, $n_1$ is a prime power, say $n_1 = p^a$ with p prime and $a \in \mathbb{N}$. If $P \in M_1$, then the projective plane $(\widetilde{\mathcal{M}}_1)_P$ associated with $(\mathcal{M}_1)_P$ is a subplane of the projective plane $\widetilde{\mathcal{M}}_P$ associated with $\mathcal{M}_P$ (this is why we required in the definition of a subplane of a Minkowski plane, that circles tangent in $\mathcal{M}_1$ are also tangent in $\mathcal{M}$). In fact $(\widetilde{\mathcal{M}}_1)_P$ is a 2-subplane of $\widetilde{\mathcal{M}}_P$ in the sense of OSTROM and WAGNER [12]. By their Theorem 6, $n = n_1^{2^g}$ for some integer g. Hence, also n is a prime power, $n = p^b$ with $b = a2^g$. Let $\Pi$ be a Sylow p-subgroup of $\Gamma_P$, $P \in M$. Let $\pi$ be an element in the centre of $\Pi$.

Since $\pi$ fixed the two ideal points corresponding to $L^+$ and $L^-$ of $M_P$, $\pi$ also fixes an affine line L of $M_P$. Suppose L intersects the ideal line of $M_P$ in a point A. Then $\Pi$ fixes A for if $A^\sigma \neq A$ for some $\sigma \in \Pi$, then $L^\sigma$ and L intersect in an affine point Q of $M_P$. Since $\Pi$ permutes the fixed objects of $\pi$, $L^\sigma$ hence Q is fixed by $\pi$. Since $\Gamma_P$, hence $\Pi$, is transitive on the $n^2$ affine points of $M_P$, every affine point of $M_P$ is fixed by $\pi$, i.e. $\pi = 1$ a contradiction. By Theorem 3 of [13] $\Gamma_{P,A}$, hence $\Pi$ is transitive on the n affine lines through A. Therefore $\pi$ fixes all lines through A, i.e. $\pi$ is an elation of $\tilde{M}_P$ with centre A and axis the ideal line of $M_P$. Suppose A is the ideal point corresponding to $L^{-\varepsilon}$ for $\varepsilon = +$ or $-$, then $\pi \in \Gamma(L^{-\varepsilon})_{[P]_\varepsilon}$. By Lemma 5, $\Gamma(L^{-\varepsilon})_{\ell,m} = 1$ for distinct lines $\ell, m \in L^\varepsilon$, so by Lemma 3, $p \leq$ order of $\pi \leq |\Gamma(L^{-\varepsilon})_{[P]_\varepsilon}| \leq 3$, i.e. $p = 3$. Also $\Gamma(L^{-\varepsilon})$ is a Frobenuis group on the (n+1) lines of $L^\varepsilon$, $\Gamma(L^{-\varepsilon}) \trianglelefteq \Gamma$ and $\Gamma$ acts two-transitively on $L^\varepsilon$, hence the Frobenius kernel of $\Gamma(L^{-\varepsilon})$ is an elementary abelian 2-group and in particular $n+1 = 2^c$ for some $c \in \mathbb{N}$. However, $n+1 = p^b + 1 = 3^{a2^g} + 1 \equiv 2(4)$ and so we have shown that A is an ideal point corresponding to a pencil with carrier P. Let T be the group of translations of $M_P$ contained in $\Gamma_P$ and for each pencil $P$ with carrier P let $T(P)$ be the group of translations of T fixing all circles of $P$. By Lemma 10 and Lemma 8(ii), $\Gamma_P$ has two orbits of length $\frac{1}{2}(n-1)$ on the pencils with carrier P. Put $x = |T(P)|$ for $P$ in the first, and $y = |T(P)|$ for $P$ in the second orbit. It follows that

$$|T| = 1 + (x-1)\cdot\tfrac{1}{2}(n-1) + (y-1)\cdot\tfrac{1}{2}(n-1) = 1 + \tfrac{1}{2}(x+y-2)(n-1), \tag{1}$$

and one of x and $y \geq p$, so $x+y \geq p+1$. Also, if s is the number of T-orbits on $M_P$,

$$s|T| = n^2. \tag{2}$$

Since $x+y \geq p+1 \geq 4$, it follows that $|T| \geq n$, hence $s \leq n$. From (1) and (2) it also follows that $s \equiv 1 (\mathrm{mod}\ \frac{1}{2}(n-1))$. Since T is not transitive on $M_P$, $s > 1$. Therefore $s = n$, $|T| = n$ and $p = 3$. We list some properties of T.

(i) As a translation group containing translations in different directions, T is elementary abelian,

(ii) $T \triangleleft \Gamma_P$,

(iii) T acts regularly on the lines of $L^{\varepsilon}\backslash\{[P]_{\varepsilon}\}$, $\varepsilon = +,-$,

(iv) the subgroups $\langle\tau\rangle$, $\tau \in T$ are in 1-1 correspondence with the $\frac{1}{2}(n-1)$ pencils with carrier P in a $\Gamma_P$-orbit: $\tau \leftrightarrow$ pencil $P$ iff centre of $\tau = P$; $\Gamma_P$ acts on this orbit as $\Gamma_P$ acts on $\{\langle\tau\rangle \mid \tau \in T\}$ by conjugation.

Take $Q \in M_P$. By Lemma 8(iii), $\Gamma_{P,Q}$ is still transitive on the pencils with carrier P in a $\Gamma_P$-orbit, so $\Gamma_{P,Q}$ acts by conjugation transitively on the subgroups $\langle\tau\rangle$, $\tau \in T$. By (ii) and (iii), T is a regular normal cubgroup of $\Gamma_P$ considered as a permutation group on $L^{+}\backslash\{[P]_{+}\}$. Since $\Gamma_{P,Q} \le \Gamma_{P,[Q]_{\varepsilon}}$, $\Gamma_{P,Q}$ acts on $L^{+}\backslash\{[P]_{+},[Q]_{+}\}$ as it does on $T\backslash\{1\}$ by conjugation. It follows that either $\Gamma_{P,Q}$ is transitive or has two orbits of length $\frac{1}{2}(n-1)$ on $L^{+}\backslash\{[P]_{+},[Q]_{+}\}$. The former alternative is impossible: an involution in the center of a Sylow 2-subgroup of $\Gamma_P$ is a homology (see the last part of the proof of Lemma 10). Therefore $\Gamma_{P,Q}$ has 2 orbits of length $\frac{1}{2}(n-1)$ on $L^{+}\backslash\{[P]_{+},[Q]_{+}\}$ and it acts on both orbits as it acts on the subgroups $\langle\tau\rangle$, $\tau \in T$ by conjugation. Let c be a circle through P and Q in the pencil $P$, where $P$ is the centre of $\langle\tau\rangle$, say. Then $\Gamma_{P,Q,c}$ fixes $P$, hence $\Gamma_{P,Q,c}$ fixes $\langle\tau\rangle$ by conjugation and therefore also two distinct lines $\ell,m \in L^{+}\backslash\{[P]_{+},[Q]_{+}\}$. Therefore also $\ell \cap c$ and $m \cap c$ are fixed by $\Gamma_{P,Q,c}$. By Lemma 11, $\Gamma_{P,Q,c}$ has a subplane $M_2$ as a set of fixed points. Let $n_2$ be the order of $M_2$ and let $c^{*}$ be the set of points left fixed by $\Gamma_{P,Q,c}$. With $B = \{c^{*\gamma} \mid \gamma \in \Gamma_c\}$ we get a 2-$(n+1,n_2+1,1)$ design on c (see [10]). The number of blocks through a point is $n/n_2 = 3^b/n_2$. Hence $n_2 = 3^d$ for some $d \in \mathbb{N}$. The total number of blocks equals $(n+1)n/(n_2+1)n = (3^b+1/3^d+1)\cdot 3^{b-d}$. Hence $\frac{b}{d} \in 2\mathbb{N}+1$. Since b is even, d is even so $10 \le n_2+1 = 3^d+1 \equiv 2 \pmod 4$. However, $\Gamma_{c^*} = N_{\Gamma_c}(\Gamma_{P,Q,c})$ is sharply 2-transitive on the $n_2+1$ points of $c^{*}$, and so $n_2+1$ is a power of 2. This was our final contradiction. □

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